Classifying Cap Obstructions of Connectivity 2

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1. Abstract

A graph G is **planar** if it can be drawn on a flat surface (a plane) in such a way that no two edges cross. Furthermore, G is an **apex graph** if it contains at most one vertex whose deletion results in a planar graph. Just as K_5 and $K_{3,3}$ are the two obstructions for planarity, it is known that the apex graphs are characterized by a finite list of obstructions. It is a long-standing open problem to determine this finite list.
We introduce the related idea of cap. A graph is **cap** if the graph is apex or has a Δ-cap, the edges of a triangle, whose deletion gives a planar graph. Jobson and Kézdy classified apex obstructions of connectivity 2 ($\kappa(G) = 2$). We will mimic their work to determine cap obstructions of connectivity 2. So far, 118 of the 133 obstructions as identified by Jobson and Kézdy have been found to also be cap obstructions of connectivity two. The question remains whether this list is complete or there are more to find.

2. Background

Definition 1. A graph G has **connectivity 2** if there exists 2 distinct vertices whose removal yields a disconnected graph (see Figure 7).

Definition 2. Let G be a graph. A **cut set** of G is a set of vertices, $S \subseteq V(G)$ such that $G - S$ is a disconnected graph.

Definition 3. A graph G is **apex** if there exists at most one vertex $v \in V(G)$ such that $G - v$ is a planar graph. Furthermore, a graph G is **2-apex** if there is at most two vertices $u, v \in V(G)$ such that $G - u, v$ is planar.

Definition 4. A graph H is a **minor** of G if H can be obtained from G through a series of edge contractions, edge deletions, or vertex deletions. Furthermore, H is a **proper minor** of G if H is a minor of G and $H \neq G$.

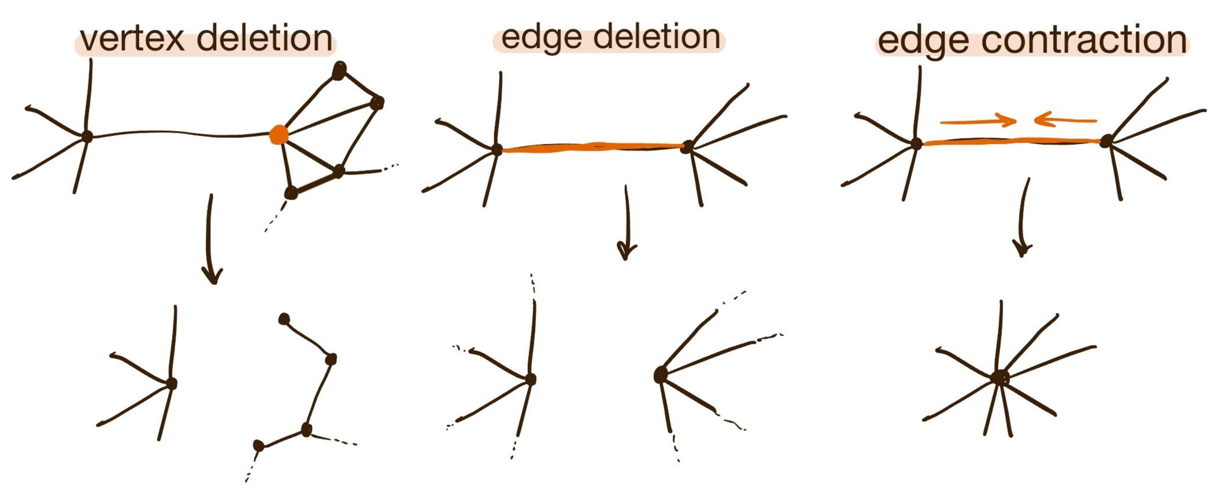

Figure 1: *Minor Moves.*

Note: a planar graph is considered apex, as deleting any vertex from a planar graph will yield a planar graph.

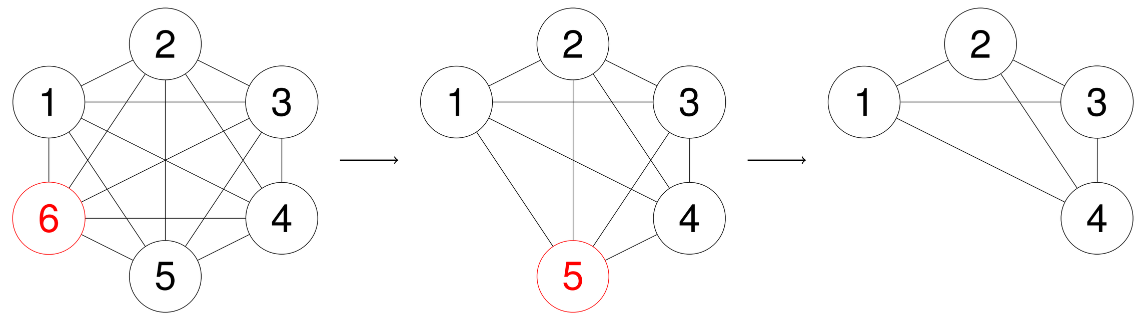

Figure 2: *Deleting two vertices from K_6 results in the planar graph K_4.*

We concern ourselves with properties that are **closed under taking minors**. For example, if G is a planar graph, no amount of minor moves will make it nonplanar. Furthermore, if G is apex it will remain apex under minor moves.

Definition 5. A graph G is **cap** if a planar graph can be obtained by removing at most one subgraph, either a vertex or the edges of a 3-cycle.

Robertson and Seymour's graph minor theorem implies that, for any **minor closed property**, there exists a finite set of **minor minimal non-members**, called the **obstruction set.**

Upshot: if we want to identify cap graphs of connectivity two, we only need to classify the finite list of obstructions.

3. Obstructions

If we were given a random graph, how would we go about determining whether it is planar?
This question was answered by Kazimierz Kuratowski in 1930 when he proved the following theorem:

Theorem 6 (Kuratowski, 1930 [2]). *A graph G is planar if and only if it contains neither a K_5 nor a $K_{3,3}$ graph as a minor.*

Why K_5 and $K_{3,3}$?
K_5 and $K_{3,3}$ are the 'smallest' nonplanar graphs in that any graph you obtain from them by deleting edges, vertices, or contracting edges is a planar graph.

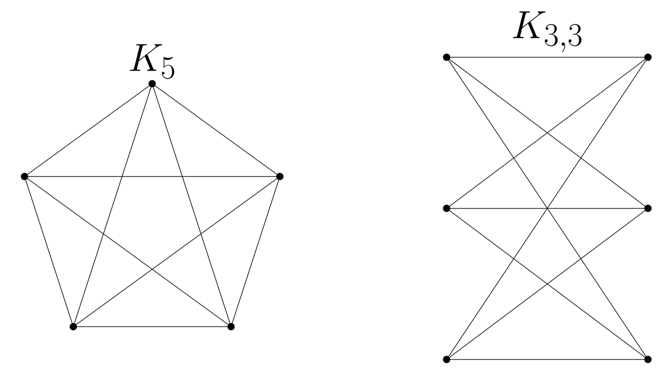

Figure 3: *Left: K_5, Right: $K_{3,3}$*

In identifying these 'smallest' graphs, Kuratowski completely classified all planar graphs. K_5 and $K_{3,3}$ form the set of obstructions for planar graphs.

Definition 7. A graph G is an **obstruction** for a property $\mathcal{P}$ if G does not have the property $\mathcal{P}$ but every proper minor of G does.

We can thus restate Kuratowski's theorem as:

Theorem 8 (Restatement of Kuratowski's Theorem)**.** *The obstructions for planarity are K_5 and $K_{3,3}$.*

This classification allows us to easily check if a given graph is planar, all we need to do is check for a K_5 or $K_{3,3}$!

4. Our Work & Motivation

Let G be a graph. G is **linklessly embeddable** if we can embed the graph in 3-space such that no non-trivial links are present. In a graph, a **link** is defined to be two or more disjoint cycles that link together like a chain. It is known that there are seven obstructions for linkless embedding: The Petersen Family.

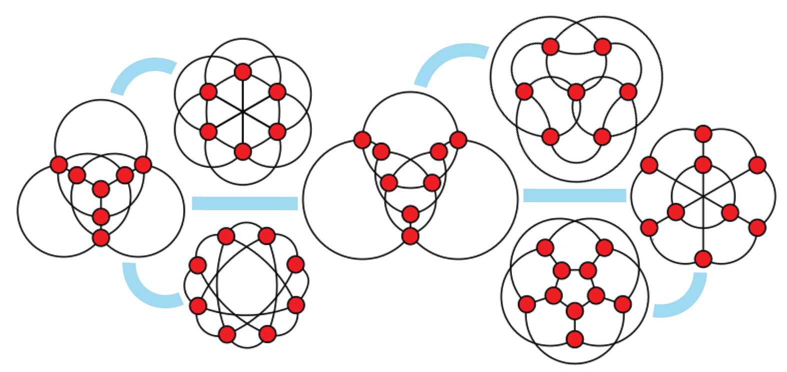

Figure 4: *The 7 Petersen graphs form the obstructions for linkless embeddings.*

Ultimately, we would like to classify the obstructions to knotless embeddings of graphs. But it's hard: there are more than 1,000 obstructions (so far!) Since 2-apex graphs have a knotless embedding, it's logical to try to classify those instead.
A problem arises! It turns out that a 2-apex graph can be made apex by performing a ΔY move.

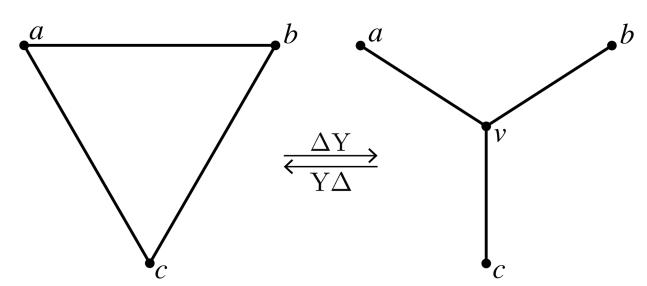

Figure 5: *A ΔY move and its counterpart, $Y\Delta$.*

We thus introduce **cap** as an analogue of apex because cap graphs are **invariant under $Y\Delta$ moves**, better reflecting the topological invariance of linkless and knotless embeddings.
Jobson and Kézdy characterized apex obstructions of connectivity two by looking at the properties of each of the two disconnected components upon deletion of the two vertices. Their classification is as follows:

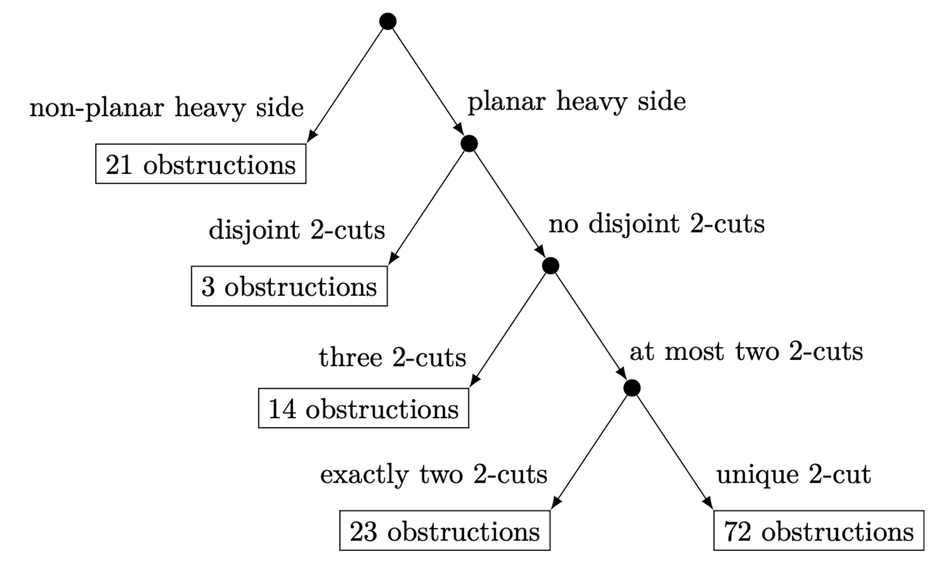

Figure 6: *The 133 apex obstructions of connectivity 2 described by Jobson and Kézdy. Retrieved from [1].*

Of the 133 apex obstructions due to Jobson and Kézdy, 118 of them are cap obstructions. The 15 that are missing can be made apex by performing a ΔY move, and thus are obstructions for apex but not cap. See below:

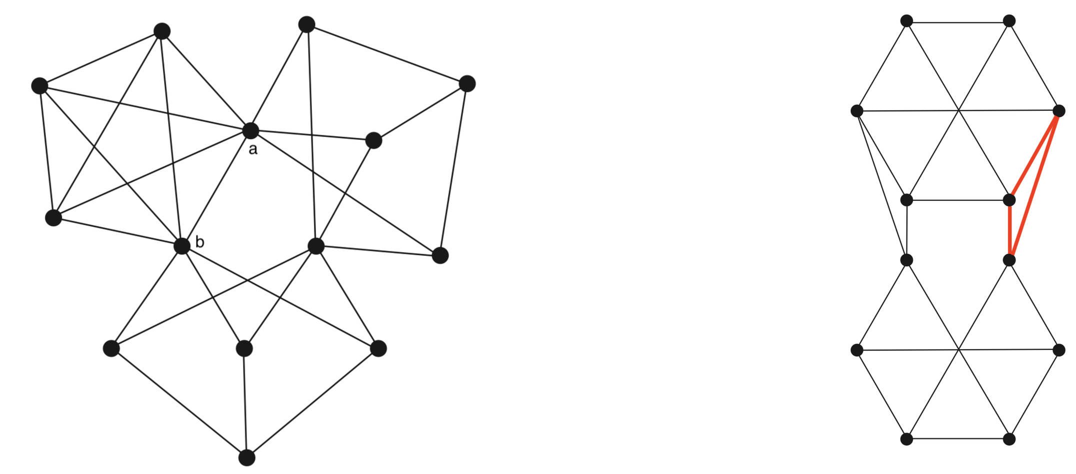

Figure 7: *Both of these graphs have connectivity 2. The left graph is an obstruction for both cap and apex. Removing the edges of the red triangle in the right graph yields a planar graph, so it is an obstruction to apex and **not** cap. Images retrieved from [3].*

Our research follows a similar classification as in Figure 6. G has connectivity 2, so let $\{a, b\} \subseteq V(G)$ be the cut set. First, we differentiate between graphs that have the edge $ab \in E(G)$ and the graphs that don't. The graphs that don't have $ab \in E(G)$ yield further classifications relating to the structure of the cut set and the overall structure of the graph.

5. Questions for Future Work

Question 1. *Is it the case that Ob(Cap) $\subseteq$ Ob(apex)? Can a counter example be found?*

Question 2. *It is known that there are no cap graphs of connectivity $\geq$ 6, how do we classify the obstructions for cap graphs of connectivity 3, 4, 5?*

Question 3. *Why does the number of obstructions jump from 2 for planar graphs, to 7 for linklessly embeddable, all the way up to the 1,000+ obstructions (so far!) for knotless? Is there a pattern in this sequence?*

References

[1] Adam S. Jobson and André E. Kézdy. All minor-minimal apex obstructions with connectivity two. *Electronic Journal of Combinatorics*, 28(1), 2021.

[2] K. Kuratowski. Sur le problème des courbes gauches en topologie. *Fundamenta Mathematicae*, 15:271–283, 1930.

[3] Max Lipton, Eoin Mackall, Thomas W. Mattman, Mike Pierce, Samantha Robinson, Jeremy Thomas, and Ilan Weinschelbaum. Six variations on a theme: almost planar graphs. *Involve, a Journal of Mathematics*, 11(3):413–448, January 2018.